Exercise 95

Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of the drug that is present at the time. Thus $C'(t) = -kC(t)$, where k is a positive number called the *elimination constant* of the drug.

(a) If C_0 is the concentration at time $t = 0$, find the concentration at time t.

(b) If the body eliminates half the drug in 30 hours, how long does it take to eliminate 90% of the drug?

Solution

Start with the differential equation for the concentration.

$$C'(t) = -kC(t)$$

Divide both sides by $C(t)$.

$$\frac{C'(t)}{C(t)} = -k$$

Rewrite the left side as a derivative of a logarithm by using the chain rule.

$$\frac{d}{dt} \ln C = -k$$

The function you take a derivative of to get $-k$ is $-kt + D$, where D is any constant.

$$\ln C = -kt + D$$

Exponentiate both sides to solve for C.

$$e^{\ln C} = e^{-kt+D}$$

$$C(t) = e^D e^{-kt}$$

Part (a)

At $t = 0$, the concentration is known to be C_0.

$$C(0) = e^D e^{-k(0)} = C_0 \quad \to \quad e^D = C_0$$

Therefore, the concentration at time t is

$$C(t) = C_0 e^{-kt}.$$

Part (b)

Use the fact that the body eliminates half the drug in 30 hours to determine k.

$$C(30) = C_0 e^{-k(30)}$$

$$\frac{C_0}{2} = C_0 e^{-30k}$$

$$\frac{1}{2} = e^{-30k}$$

$$\ln \frac{1}{2} = \ln e^{-30k}$$

$$-\ln 2 = (-30k) \ln e$$

$$k = \frac{\ln 2}{30}$$

As a result, the concentration after t hours is

$$\begin{aligned} C(t) &= C_0 e^{-\left(\frac{\ln 2}{30}\right)t} \\ &= C_0 e^{\ln 2^{-t/30}} \\ &= C_0 (2)^{-t/30}. \end{aligned}$$

To determine how long it takes to eliminate 90% of the drug, set $C(t) = 0.1C_0$ and solve the equation for t.

$$0.1C_0 = C_0 (2)^{-t/30}$$

$$0.1 = 2^{-t/30}$$

$$\ln 0.1 = \ln 2^{-t/30}$$

$$-\ln 10 = \left(-\frac{t}{30}\right) \ln 2$$

$$t = \frac{30 \ln 10}{\ln 2} \approx 99.6578 \text{ hours}$$